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Radiative corrections to the $K_{e3}^{\pm}$ decay revised

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Abstract. We consider the lowest order radiative corrections for the decay $K^{\pm} \to \pi^0 e^{\pm} \nu$, usually referred to as $K_{e3}^{\pm}$ decay. This decay is the best way to extract the value of the V_{us} element of the CKM matrix. The radiative corrections become crucial if one wants a precise value of V_{us}. The existing calculations were performed in the late 60's and are in disagreement. The calculation by Ginsberg turns out to be ultraviolet cutoff sensitive. The necessity of precise knowledge of V_{us} and the contradiction between the existing results constitute the motivation of our paper.
We remove the ultraviolet cutoff dependence by using Sirlin's prescription; we set it equal to the W mass. We establish the whole character of the small lepton mass dependence based on the renormalization group approach. In this way we can provide a simple explanation of Kinoshita–Lee–Nauenberg cancellation of singularities in the lepton mass terms in the total width and pion spectrum. We give an explicit evaluation of the structure-dependent photon emission based on ChPT in the lowest order. We estimate the accuracy of our results to be at the level of 1%. The corrected total width is $\Gamma = \Gamma_0(1+\delta)$ with $\delta = 0.02 \pm 0.0002$. Using the form factor value $f_+(0) = 0.9842 \pm 0.0084$ calculated by Cirigliano et al. leads to $|V_{us}| = 0.2172 \pm 0.0055$.

1 Motivation

For corrections due to virtual photons, see Fig. 1; for corrections due to real photons, see Fig. 2.

The K_{e3} decay is important since it is the cleanest way to measure the V_{us} matrix element of the CKM matrix. If one uses the current values for V_{ud}, V_{us}, and V_{ub} taken from the PDG then $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2$ misses unity by 2.2 standard deviations, which contradicts the unitarity of the CKM matrix and might indicate physics beyond the standard model. The uncertainty brought to the above expression by V_{us} is about the same as the uncertainty that comes from V_{ud}. Therefore, reducing the error in the V_{us} matrix element would substantially reduce the error in the whole unitarity equation. Reliable radiative corrections, potentially of the order of a few percent, are necessary to extract the V_{us} matrix element from the K_{e3} decay width with high precision.

Calculations of the radiative corrections to the K_{e3} decay were performed independently by Ginsberg and Becherrawy in the late 60's [2,1]. Their results for corrections to the decay rate, Dalitz plot, pion and positron spectra disagree, in some places quite sharply; for example Ginsberg's correction to the decay rate is -0.45%, while that of Becherrawy is -2% (corresponding to corrections to the total width Γ of 0.45 and 2 respectively). We have decided to perform a new calculation since the results of the experiments will become available soon and we want to explore the causes of the discrepancies in the previous calculations. Recently a revision of Ginsberg's paper, with a numerical estimation of the radiation corrections [14] was published.

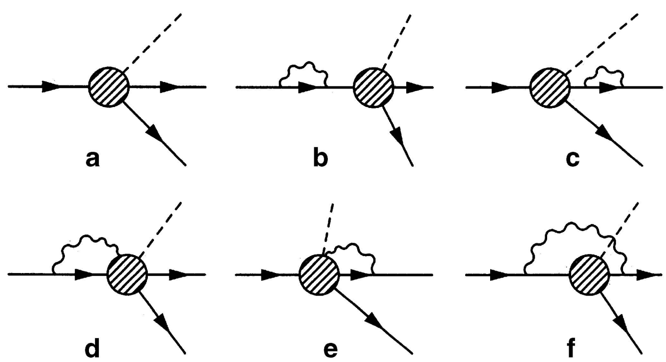

Fig. 1a–f. Virtual photons

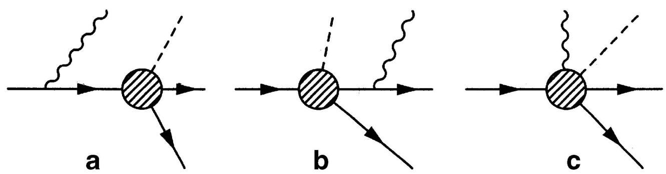

Fig. 2a–c. Real photons

Our paper is organized as follows. The introduction (Sect. 2) is devoted to a short review of the kinematics of the elastic decay process (without emission of a real photon). In Sect. 3 we face the results concerning the virtual and soft real photons' emission contribution to the differential width. In Sect. 4 we consider the hard photon emission including both the inner bremsstrahlung (IB) and the structure-dependent (SD) contributions and derive an expression for the differential width by starting with the Born width and adding the known structure functions in the leading logarithmical approximation (the so-called Drell–Yan picture of the process). We give the explicit expressions for the non-leading contributions. In Sect. 5, we summarize our results and compare them with those in the previously published papers.

Appendix A contains the details of the calculations of virtual and real soft photon emission.

Appendix B contains the details of the description of hard photon emission both by the IB and the SD mechanism. Our approach to the study of hard photon emission differs technically from the ones used in [1,2].

Appendix C contains the explicit formulae for the description of SD emission including the interference of IB and SD amplitudes.

Appendix D is devoted to an analysis of the Dalitz-plot distribution and the properties of the Drell–Yan conversion mentioned above.

Appendix E contains a list of the formulae used for the numerical integration.

Appendix F contains the details of the kinematics of radiative kaon decay and, besides the analysis of relations of our paper and [2], technical approaches.

In Tables 1 and 2 and Figs. 3, 4, 5 and 6 the result of the numerical estimation of the Born values and the correction to the Dalitz-plot distribution and pion and positron spectra are given.

2 Introduction

The lowest order perturbation theory (PT) matrix element of the process $K^+(p) \to \pi^0(p') + e^+(p_e) + \nu(p_\nu)$ has the form

$$M = \frac{G_\mathrm{F}}{\sqrt{2}} V^*_{us} F_\nu(t) \bar{u}(p_\nu)\gamma_\nu(1+\gamma_5)v(p_e), \tag{1}$$

where $F_\nu(t) = (1/2^{1/2})(p+p')_\nu f_+(t)$. The Dalitz-plot density which takes into account the radiative corrections (RC) of the lowest order PT is

$$\begin{aligned}\frac{\mathrm{d}^2\Gamma}{\mathrm{d}y\mathrm{d}z} &= \mathcal{C} a_0(y,z)(1+\delta(y,z))\left(1+\lambda_+\frac{t}{m_\pi^2}\right)^2 \\ &= \frac{\mathrm{d}^2\Gamma_0(y,z)}{\mathrm{d}y\mathrm{d}z}(1+\delta(y,z)), \\ \Gamma_0 &= \int \frac{\mathrm{d}^2\Gamma_0}{\mathrm{d}y\mathrm{d}z}\mathrm{d}y\mathrm{d}z, \quad \Gamma = \int \frac{\mathrm{d}^2\Gamma}{\mathrm{d}y\mathrm{d}z}\mathrm{d}y\mathrm{d}z,\end{aligned} \tag{2}$$

where the momentum transfer squared between kaon and pion is

$$t = (p-p')^2 = M_K^2(1+r_\pi - z) = M_K^2 R(z).$$

We accept here the following form for the strong interactions induced form factor $f_+(t)$:

$$f_+(t) = f_+(0)\left(1+\lambda_+\frac{t}{m_\pi^2}\right), \tag{3}$$

according to PDG $\lambda_+ = 0.0276 \pm 0.0021$. From now on we will use M^2 instead of M_K^2. We define

$$\mathcal{C} = \frac{M^5 G_\mathrm{F}^2 |V_{us}|^2}{64\pi^3}|f_+(0)|^2 \tag{4}$$

and

$$a_0(y,z) = (z+y-1)(1-y) - r_\pi + O(r_e). \tag{5}$$

Here we follow the notation of [4]:

$$r_e \equiv m_e^2/M^2, \quad r_\pi \equiv m_\pi^2/M^2, \tag{6}$$

where m_e, m_π, and M_K are the masses of electron, pion, and kaon; two convenient kinematical variables are

$$y \equiv 2pp_e/M^2, \quad z \equiv 2pp'/M^2. \tag{7}$$

In the kaon's rest frame, which we will imply throughout the paper, y and z are the energy fractions of the positron and pion:

$$y = 2E_e/M, \quad z = 2E_\pi/M. \tag{8}$$

The region of the y,z-plane where $a_0(y,z) > 0$ will be named region D. Later, when dealing with real photons we will also use

$$x = 2\omega/M, \tag{9}$$

with ω the real photon energy.

The physical region for y and z (further called the D region) is [4]

$$\begin{aligned} 2\sqrt{r_e} &\le y \le 1 + r_e - r_\pi, \\ F_1(y) - F_2(y) &\le z \le F_1(y) + F_2(y), \\ F_1(y) &= (2-y) \\ &\quad \times (1+r_e+r_\pi-y)/[2(1+r_e-y)], \\ F_2(y) &= \sqrt{y^2-4r_e} \\ &\quad \times (1+r_e-r_\pi-y)/[2(1+r_e-y)], \end{aligned} \tag{10}$$

or, equivalently,

$$\begin{aligned} 2\sqrt{r_\pi} &\le z \le 1 + r_\pi - r_e, \\ F_3(z) - F_4(z) &\le y \le F_3(z) + F_4(z), \\ F_3(z) &= (2-z) \\ &\quad \times (1+r_\pi+r_e-z)/[2(1+r_\pi-z)], \\ F_4(z) &= \sqrt{z^2-4r_\pi} \\ &\quad \times (1+r_\pi-r_e-z)/[2(1+r_\pi-z)]. \end{aligned} \tag{11}$$

For our aims we use the simplified form of the physical region (omitting the terms of the order of r_e):

$$2\sqrt{r_e} \le y \le 1 - r_\pi, \quad c(y) \le z \le 1 + r_\pi, \tag{12}$$

with

$$c(y) = 1 - y + \frac{r_\pi}{1-y},$$

or

$$2\sqrt{r_\pi} \le z \le 1 + r_\pi, \quad b_-(z) \le y \le b(z), \tag{13}$$

with

$$b_-(z) = 1 - \frac{1}{2}\left(z + \sqrt{z^2 - 4r_\pi}\right),$$
$$b(z) = 1 - \frac{1}{2}\left(z - \sqrt{z^2 - 4r_\pi}\right).$$

For definiteness we give here the numerical value for the Born total width. It is

$$\frac{G_F^2 M_K^5 |V_{us} f_+(0)|^2}{64\pi^3} \int \mathrm{d}y \int \mathrm{d}z a_0(y,z) \left(1 + \lambda_+ \frac{t}{m_\pi^2}\right)^2$$
$$= 5.36 |V_{us} f_+(0)|^2 \times 10^{-14}\,\mathrm{MeV}. \tag{14}$$

Comparing this value with the PDG result, $(\Gamma_{K+e3})_{\exp} = (2.56 \pm 0.03) \times 10^{-15}$ MeV, we conclude

$$(V_{us} f_+(0))|_{\alpha=0} = 0.218 \pm 0.002. \tag{15}$$

3 Virtual and soft real photon emission

Taking into account the accuracy level of 0.1% for the determination of ρ/ρ_0 we will drop terms of order r_e. We will distinguish three kinds of contributions to δ: from the emission of virtual, soft real, and hard real photons in the rest frame of the kaon: $\delta = \delta_\mathrm{V} + \delta_\mathrm{S} + \delta_\mathrm{H}$. A standard calculation (see Appendix A for details) allows one to obtain the following contributions: from the soft real photons

$$\delta_\mathrm{S} = \frac{\alpha}{\pi}\left\{(L_e - 2)\ln\frac{2\Delta\epsilon}{\lambda} + \frac{1}{2}L_e - \frac{1}{4}L_e^2 + 1 - \frac{\pi^2}{6}\right\}$$
$$\times (1 + O(r_e)), \tag{16}$$

from the virtual photons $\delta_\mathrm{V} = \delta_\mathrm{C} + \delta_\mathrm{PLM}$ that make up a charged fermion's renormalization, δ_C (throughout this paper we use the Feynman gauge):

$$\delta_\mathrm{C} = \frac{\alpha}{2\pi}\left\{\left[-\frac{1}{2}L_\Lambda + \frac{3}{2}\ln r_e + \ln\frac{M^2}{\lambda^2} - \frac{9}{4}\right]\right.$$
$$\left. + \left[L_\Lambda + \ln\frac{M^2}{\lambda^2} - \frac{3}{4}\right]\right\}; \tag{17}$$

here $L_\Lambda = \ln(\Lambda^2/M^2)$, Λ is the ultraviolet momentum cutoff, the first term in the curly braces comes from the positron, the second one from the kaon; and for the diagram in Fig. 1f in the point-like meson (PLM) approximation, δ_PLM:

$$\delta_\mathrm{PLM} = -\frac{\alpha}{2\pi}\left\{ - L_\Lambda - \frac{1}{2}\ln^2 r_e - 2L_e + \ln\frac{M^2}{\lambda^2}L_e - 1\right.$$
$$\left. + 2\ln^2 y + 2\ln y + 2\mathrm{Li}_2(1-y)\right\}. \tag{18}$$

When these contributions are grouped all together the dependence on λ (the fictitious "photon mass") disappears. According to Sirlin's prescription [7] we set $\Lambda = M_W$. The result can be written in the form

$$1 + \delta_\mathrm{S} + \delta_\mathrm{C} + \delta_\mathrm{PLM}$$
$$= S_W\left[1 + \frac{\alpha}{\pi}\left[(L_e - 1)\left(\ln\Delta + \frac{3}{4}\right) - \ln\Delta\right.\right.$$
$$\left.\left. - \frac{\pi^2}{6} + \frac{3}{4} - \mathrm{Li}_2(1-y) - \frac{3}{2}\ln y\right]\right],$$
$$S_W = 1 + \frac{3\alpha}{4\pi}L_W. \tag{19}$$

In the above equations $L_e = 2\ln y + \ln(1/r_e)$, and $L_W = \ln(M_W^2/M^2)$; M_W is the mass of $W^\pm$, $\Delta = \Delta\epsilon/E_e$, and $\Delta\epsilon$ is the maximal energy (in the rest frame of the kaon) of a real soft photon. We imply $\Delta\epsilon \ll M/2$. For the details of (16), (17), (18), and (19) see Appendix A.

The contribution from soft photon emission from the structure-dependent part (such as for example, interaction with resonances and intermediate $W^\pm$) is small, of the order

$$\frac{\alpha}{\pi}\frac{\Delta\epsilon}{M} \ll 1,$$

and thus is also neglected.

4 Hard photon emission. Structure function approach

Next we need to calculate contributions from hard photons. We have to distinguish between inner bremsstrahlung (IB) and the structure-dependent (SD) contributions: $\delta_\mathrm{H} = \delta_\mathrm{IB} + \delta_\mathrm{int} + \delta_\mathrm{SD}$, where δ_int is the interference term between the two. The terms δ_int and δ_SD are considered in the framework of the chiral perturbation theory (ChPT) to the orders of (p^2) and (p^4) and find their contribution to be at the level of 0.2% (see Appendix C). We have

$$\delta_\mathrm{H} = \frac{\alpha}{2\pi a_0(y,z)}$$
$$\times\left\{(L_e - 1)\left(\Psi(y,z) - a_0(y,z)\left(2\ln\Delta + \frac{3}{2}\right)\right)\right.$$
$$\left. - 2a_0(y,z)\ln\frac{b(z) - y}{y\Delta}\right\} + \delta^\mathrm{hard}, \tag{20}$$

with δ^hard given below.

Extracting the short-distances contributions in the form of the replacement $\mathcal{C} \to \mathcal{C}S_W$ it is useful to split δ (see (2)) in the form

$$\delta(y,z) = \delta_\mathrm{L} + \delta_\mathrm{NL}, \tag{21}$$

where δ_L is the leading order contribution; it contains the "large logarithm" L_e, and δ_NL is the non-leading contribution; it contains the rest of the terms.

δ_{L} contains terms from δ_{C}, δ_{S}, δ_{PLM}, and the contribution from the collinear configuration of hard IB emission (in the collinear configuration the angle between the positron and the emitted photon is small). δ_{L} turns out to be

$$\delta_{\mathrm{L}} = \frac{\alpha(L_e-1)}{2\pi a_0(y,z)}\Psi(y,z). \tag{22}$$

First we note that the kinematics of hard photon emission does not coincide with the elastic process (region D, the strictly allowed boundaries of the Dalitz plot). In hard photon emission an additional region in the y,z-plane, namely $y < b_-(z)$ appears. The nature of this phenomenon is the same as the known phenomenon of the radiative tail in the process of hadron production at colliding e^+e^- beams.

The quantity $\Psi(y,z)$ has a different form for region D and outside it:

$$\Psi(y,z) = \Psi_>(y,z), \quad z > c(y), \quad 2\sqrt{r_e} < y < 1 - r_\pi, \tag{23}$$

and

$$\Psi(y,z) = \Psi_<(y,z), \quad z < c(y), \quad 2\sqrt{r_e} < y < 1 - \sqrt{r_\pi}. \tag{24}$$

$\Psi_<(y,z) = 0$ when $y > 1 - r_\pi^{1/2}$. The functions $\Psi_<, \Psi_>$ are studied in Appendix D.

δ_{NL} contains contributions from δ_{C}, δ_{PLM}, from the SD part of hard photons and from the interference term of the SD and IB parts of the hard radiation. We have

$$\delta_{\mathrm{NL}} = \frac{\alpha}{\pi}\eta(z,y), \tag{25}$$

where

$$\frac{\alpha}{\pi}\eta(y,z) = \delta^{\mathrm{hard}} + \frac{\alpha}{\pi}\left[\frac{3}{4} - \frac{\pi^2}{6} - \mathrm{Li}_2(1-y) - \frac{3}{2}\ln y \right.$$
$$\left. - \ln((b(z)-y)/y)\right], \tag{26}$$

and for the case when the variables y,z are inside the D region:

$$\delta^{\mathrm{hard}} = \frac{\alpha}{2\pi a_0(y,z)} Z_2(y,z); \tag{27}$$
$$Z_2(y,z) = -2\mathrm{Rphot}_{1D}(y,z) + \mathrm{Rphot}_{2D}(y,z) + \int\limits_0^{b(z)-y} \mathrm{d}x \mathcal{J}(x,y,z). \tag{28}$$

Explicit expressions for $\mathrm{Rphot}_{1,2}$ and $\mathcal{J}$ are given in Appendix B. The Born value and the correction to the Dalitz-plot distribution $\Delta(y,z) = \delta(y,z)a_0(y,z)$ is illustrated in Tables 1 and 2.

We see that the leading contribution from virtual and soft photon emission is associated with the so-called δ-part of the evolution equation kernel:

$$(\delta_{\mathrm{C}} + \delta_{\mathrm{S}} + \delta_{\mathrm{PLM}})^{\mathrm{leading}} = \frac{\alpha}{2\pi}(L_e-1)\int \frac{a_0(t,z)}{a_0(y,z)} P_\delta^{(1)}\left(\frac{y}{t}\right)\frac{\mathrm{d}t}{t}, \tag{29}$$

where

$$P_\delta^{(1)}(t) = \delta(1-t)\left(2\ln\Delta + \frac{3}{2}\right). \tag{30}$$

The contribution of the hard photon kinematics in the leading order can be found with the method of quasi-real electrons [10] as a convolution of the Born approximation with the θ-part of the evolution equation kernel $P_\theta(z)$:

$$\delta_{\mathrm{H}}^{\mathrm{leading}} \sim \frac{\alpha}{2\pi}(L_e-1)\int \frac{\mathrm{d}t}{t}\frac{a_0(t,z)}{a_0(y,z)} P_\theta^{(1)}\left(\frac{y}{t}\right), \tag{31}$$

where

$$P_\theta^{(1)}(z) = \frac{1+z^2}{1-z}\theta(1-z-\Delta). \tag{32}$$

In such a way the whole leading order contribution can be expressed in terms of the convolution of the width in the Born approximation with the whole kernel of the evolution equation:

$$P^{(1)}(z) = \lim_{\Delta\to 0}\left(P_\delta^{(1)}(z) + P_\theta^{(1)}(z)\right). \tag{33}$$

This approach can be extended to the use of non-singlet structure functions $D(t,y)$ [9]:

$$\mathrm{d}\Gamma^{\mathrm{LO}}(y,z) = \int\limits_{\max[y,b_-(z)]}^{b(z)} \frac{\mathrm{d}t}{t}\mathrm{d}\Gamma_0(t,z)\,D\left(\frac{y}{t},L_e\right),$$
$$t = x + y,$$
$$D(z,L) = \delta(1-z) + \frac{\alpha}{2\pi}LP^{(1)}(z) + \frac{1}{2!}\left(\frac{\alpha L}{2\pi}\right)^2 P^{(2)}(z) + \ldots, \tag{34}$$
$$P^{(i)}(z) = \int\limits_z^1 \frac{\mathrm{d}x}{x}P^{(1)}(x)P^{(i-1)}\left(\frac{z}{x}\right), \quad i = 2,3,\ldots$$

One can check the validity of the useful relation

$$\int\limits_0^1 \mathrm{d}z P^{(1)}(z) = 0. \tag{35}$$

The above makes it easy to see that in the limit $m_e \to 0$ terms that contain m_e do not contribute to the total width in correspondence with the Kinoshita–Lee–Nauenberg (KLN) theorem [12] as well as with results of Ginsberg [2]. Keeping in mind the representation

$$\Psi(y,z) = \int\limits_{\max[y,b_-(z)]}^{b(z)} \frac{\mathrm{d}t}{t}a_0(t,z)P^{(1)}\left(\frac{y}{t}\right), \tag{36}$$

one can get convinced (see Appendix D) that the leading logarithmical contribution to the total width as well as the one to the pion spectrum is zero due to

$$\int\limits_{2\sqrt{r_\pi}}^{1+r_\pi} \mathrm{d}z \int\limits_0^{b(z)} \mathrm{d}y \Psi(y,z) = 0. \quad (37)$$

Using the general properties of the evolution equations kernels, (34), one can see that KLN cancellation will take place in all orders of the perturbation theory. The spectra in Born approximation are (we omit terms $O(r_e) \sim 10^{-6}$) for the pion

$$\frac{1}{\mathcal{C}} \frac{\mathrm{d}\Gamma_0}{\mathrm{d}z} = \phi_0(z),$$
$$\phi_0(z) = \left(1 + \frac{\lambda_+}{r_\pi} R(z)\right)^2 \int\limits_{b_-(z)}^{b(z)} \mathrm{d}y a_0(y,z)$$
$$= \left(1 + \frac{\lambda_+}{r_\pi} R(z)\right)^2 \frac{1}{6} \left(z^2 - 4r_\pi\right)^{3/2}, \quad (38)$$

and for the positron

$$f(y) = \frac{1}{\mathcal{C}} \frac{\mathrm{d}\Gamma_0}{\mathrm{d}y} f_0(y) \left[1 + \frac{2}{3} \left(\frac{\lambda_+}{r_\pi}\right) \frac{y(1 - r_\pi - y)}{1 - y} + \frac{1}{6} \left(\frac{\lambda_+}{r_\pi}\right)^2 \frac{y^2(1 - r_\pi - y)^2}{(1 - y)^2}\right], \quad (39)$$

$$f_0(y) = \frac{y^2(1 - r_\pi - y)^2}{2(1 - y)}. \quad (40)$$

The corrected pion spectrum in the inclusive set-up of the experiment when integrating over the whole region for y $(0 < y < b(z))$ has the form $\phi_0(z) + (\alpha/\pi)\phi_1(z)$ with

$$\phi_1(z) = \left(1 + \frac{\lambda_+}{r_\pi} R(z)\right)^2 \times \left[\int\limits_0^{b_-(z)} \mathrm{d}y \left[\Psi_<(y,z) \ln y - a_0(y,z) \ln \frac{b(z) - y}{b_-(z) - y} + \frac{1}{2} \tilde{Z}_2(y,z)\right] + \int\limits_{b_-(z)}^{b(z)} \mathrm{d}y \left[\Psi_>(y,z) \ln y + a_0(y,z) Z_1(y,z) + \frac{1}{2} Z_2(y,z)\right]\right]; \quad (41)$$

the quantities $Z_1, \tilde{Z}_2$ are explained in Appendix E. This function does not depend on $\ln(1/r_e)$. The pion spectrum in the exclusive set-up (y, z in the region D) will depend on L_e. Its expression is given in Appendix E.

The numerical estimation of the pion spectrum is illustrated in Figs. 3 and 5.

The inclusive positron spectrum with the correction of the lowest order is $f(y) + (\alpha/\pi) f_1(y)$ with $f(y)$ given above and

$$f_1(y) = \frac{1}{2} (L_e - 1) I(y) - \int\limits_{c(y)}^{1+r_\pi} a_0(y,z) \left(1 + \frac{\lambda_+}{r_\pi} R(z)\right)^2 \ln((b(z) - y)/y) \mathrm{d}z$$
$$+ \left(\frac{3}{4} - \frac{\pi^2}{6} - \frac{3}{2} \ln y - \mathrm{Li}_2(1 - y)\right) f(y) + \frac{1}{2} \int\limits_{c(y)}^{1+r_\pi} Z_2(y,z) \left(1 + \frac{\lambda_+}{r_\pi} R(z)\right)^2 \mathrm{d}z$$
$$+ \theta(1 - \sqrt{r_\pi} - y) \int\limits_{2\sqrt{r_\pi}}^{c(y)} \mathrm{d}z \left(1 + \frac{\lambda_+}{r_\pi} R(z)\right)^2 \times \left[(1/2)\tilde{Z}_2 - a_0(y,z) \ln \frac{b(z) - y}{b_-(z) - y}\right], \quad (42)$$

with

$$I(y) = j_0(y) + \left(\frac{\lambda_+}{r_\pi}\right) j_1(y) + \left(\frac{\lambda_+}{r_\pi}\right)^2 j_2(y), \quad (43)$$

$$j_0(y) = \int\limits_y^{1-r_\pi} \frac{\mathrm{d}t}{t} \int\limits_{c(t)}^{1+r_\pi} \mathrm{d}z a_0(t,z) P^{(1)}\left(\frac{y}{t}\right)$$
$$= \left(2 \ln \frac{1 - r_\pi - y}{y} + \frac{3}{2}\right) f_0(y) + \frac{r_\pi^2 (1 + y^2)}{2(1 - y)} \ln \frac{1 - y}{r_\pi} + \frac{1}{12}(1 - r_\pi - y) \times [1 - 5r_\pi - 2r_\pi^2 + y(4 - 13r_\pi) - 17y^2]; \quad (44)$$

explicit expressions for $j_1(y)$ and $j_2(y)$ are given in Appendix D.

Numerical estimation of positron spectrum is illustrated in Figs. 3 and 6.

One can check the fulfillment of KLN cancellation of singular terms in the limit $m_e \to 0$ for the total width correction: $\int_0^{1-r_\pi} I(y)\mathrm{d}y = 0$. The expression for $j_0(y)$ agrees with A(2) from the paper of Ginsberg in the year 1966 [2].

We put here the general expression for the differential width of hard photon emission, which might be useful for the construction of Monte Carlo simulation of real photon emission in K_{e3}:

$$\mathrm{d}\Gamma_\gamma^{\mathrm{hard}} = \mathrm{d}\Gamma_0 \frac{\alpha}{2\pi} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}O_\gamma}{2\pi a_0(y,z)} T, \quad (45)$$

with

$$x = \frac{2\omega}{M} > \frac{2\Delta\epsilon}{M} = y \frac{\Delta\epsilon}{E_e}, \quad \frac{\Delta\epsilon}{E_e} \ll 1; \quad (46)$$

and $\mathrm{d}O_\gamma$ is an element of the photon's solid angle. The quantity T is explained in Appendix B.

For soft photon emission we have

$$\mathrm{d}\Gamma_\gamma^{\mathrm{soft}} = \mathrm{d}\Gamma_0 \frac{\alpha}{2\pi}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}O_\gamma}{2\pi} \times \left[-1 - \frac{r_e}{(1-\beta_e C_e)^2} + \frac{y}{1-\beta_e C_e}\right], \quad x < y\frac{\Delta\epsilon}{E_e}. \tag{47}$$

Integrating over angles within the phase volume of the hard photon we obtain the spectral distribution of radiative kaon decay:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Gamma_0 \mathrm{d}x} = \frac{\alpha}{2\pi}\frac{1}{a_0(y,z)} \times \left[\frac{a_0(x+y,z)}{x(x+y)^2}\left((y^2+(x+y)^2)(L_e-1)+x^2\right) - \frac{2}{x}a_0(y,z) - 2\left(\frac{R(z)}{x+y} - y\right) + \mathcal{J}(x,y,z)\right],$$
$$y\Delta < x < b(z) - y, \quad \Delta = \frac{\Delta\epsilon}{E_e} \ll 1. \tag{48}$$

5 Discussion

The structure-dependent contribution to the emission of virtual photons (see Fig. 1d,e) can be interpreted as a correction to the strong form factor of the $K\pi$ transition, $f_+(t)$. We assume that this form factor can be extracted from experiment and thus do not consider it. The problem of the calculation of RC to K_{e3} and especially the form factors in the framework of CHpT with virtual photons was considered in a recent paper [14].

As in paper [2] we assume a phenomenological form for the hadronic contribution to the K–π vertex, but here we use explicitly the dependence of the form factor in the form

$$f_+(t) = f_+(0)\left(1 + \frac{\lambda_+}{r_\pi}R(z)\right). \tag{49}$$

We assume that the effect of higher order ChPT as well as RC to the form factors can be taken as a multiplicative scaling factor for $f_+(0)$, which we take from a recent paper [14].

We assume an experiment in which only one positron in the final state is present, but place no limits on the number of photons. The ratio of the LO contributions to the Born contribution in the first order is a few percent, and for the second order it is about

$$\left(\frac{\alpha L_e}{2\pi}\right)^2 \le 0.03\%. \tag{50}$$

Due to the non-definite sign structure of the leading logarithm contribution (see (22)) there are regions in the kinematically allowed area where $|\Psi(y,z)|$ is close to zero. In these regions the non-leading contributions become dominant.

The contribution of the channels with more than one charged lepton in the final state as well as the vacuum polarization effects in higher orders may be taken into account by introducing the singlet contribution to the structure functions. The effect will be at the level of 0.03% and we omit them within the precision of our calculation.

The contribution of the $O(p^4)$ terms [5] turns out to be small. Indeed, one can see that they are of the order $O(\alpha L_9^r, (\bar{p}/\Lambda_c)^2) \le O(10^{-2}\%)$, $\Lambda_c = 4\pi F_\pi \approx 1.2\,\mathrm{GeV}$ ($F_\pi = 93\,\mathrm{MeV}$ is the pion life time constant), where $\bar{p}$ is the characteristic momentum of a final particle in the given reaction, $\bar{p}^2 \le M^2/16 \sim F_\pi^2$. So the terms of the orders $O(p^4)$ and $O(p^6)$ can be omitted within the accuracy of $O\left((\alpha/\pi)\times 10^{-2}\right) \le O\left(10^{-4}\right)$.

Our main results are given in (2), (21), (22), (26)–(28) for the Dalitz-plot distribution; (37)–(40) for the pion and positron spectra; (46) for hard photon emission; (52) for the value $|V_{us}|$, in the tables and figures. The accuracy of these formulas is determined by the following:

(1) we do not account higher order terms in PT, the ones of the order of $(\alpha L_e/\pi)^n, n \ge 2$ which is smaller than 0.03%;
(2) the structure-dependent real hard photon emission contribution to RC we estimate to be at the level of 0.0005;
(3) higher order CHPT contributions to the structure-dependent part are of the order 0.05% [4,5].

All the percentages are taken with respect to the Born width. All together we believe the accuracy of the results to be at the level of 0.01. So the final result of our calculation may be written in the form

$$\frac{\Gamma}{\Gamma_0} = (1 + \delta(1 \pm 0.01)); \tag{51}$$

the terms on the r.h.s. are given in (21), (22) and (25).

Here is the list of improvements comparing with the older calculations [1,2]:

(1) we eliminate the ultraviolet cutoff dependence by choosing $\Lambda = M_W$;
(2) we describe the dependence on the lepton mass logarithm L_e in all orders of the perturbation theory and explain why the correction to the total width does not depend on m_e;
(3) we treat the strong interaction effects by means of CHPT in its lowest order $O(P^2)$ and show that the next order contribution is small;
(4) we give explicit formulas for the total differential cross section and explicit results for the corrections to the Dalitz plot and particle spectra that might be used in an experimental analysis.

In the papers of Ginsberg the structure-dependent emission was not considered. Becherrawy, on the other hand, did include it, and this will give rise to differences in the Dalitz plot. In addition, Ginsberg used the proton mass as the momentum cutoff parameter.

We do not consider the evolution of coupling constant effects in the regions of virtual photon momentum modulo square from the quantities of order M_ρ^2 up to M_Z^2, which can be taken into account [11] (and for details see [14]) by replacing the quantity S_W by the $S_{\mathrm{EW}} = 1 + (\alpha/\pi)\ln(M_Z^2/M_\rho^2) = 1.0232$. Taking this replacement into

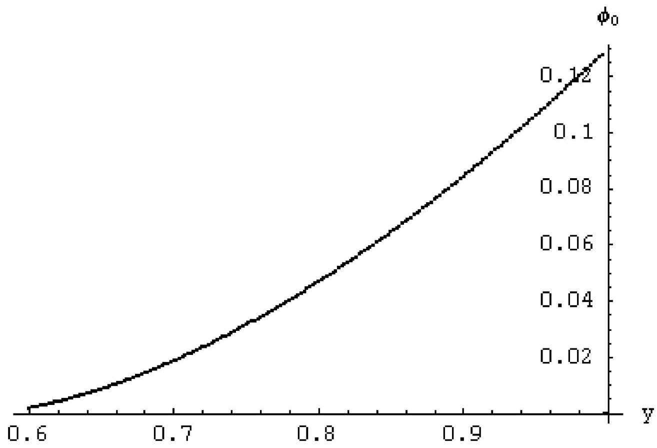

Fig. 3. Pion spectrum in Born approximation, $\phi_0(z)$ (see (39))

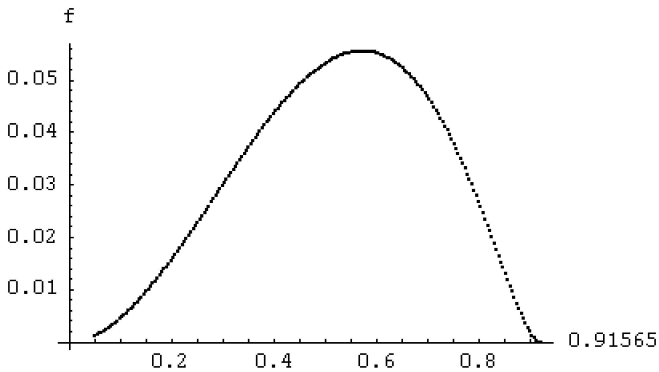

Fig. 4. Positron spectrum in Born approximation, $f(y)$ (see (39))

account our result for the correction to the total width is

$$\frac{\Gamma}{\Gamma_0} = 1 + \delta = 1.02, \tag{52}$$

which results in

$$|V_{us} f_+(0)| = 0.214 \pm 0.002. \tag{53}$$

So the correction to the total width is +2% while Ginsberg's result is −0.45% and Becherrawy's result is −2%. Neither Ginsberg nor Becherrawy used the factor $S_{\rm EW}$, and this factor (1.023) accounts for most of the difference between Ginsberg's and our result. Electromagnetic corrections become negative and have an order of 10^{-3}. The effect of the SD part, which Ginsberg did not consider, is small: of the order of 0.1%.

We use the value of the form factor $f_+(0) = 0.9842 \pm 0.0084$ calculated in [14] in the framework of ChPT, including virtual photonic loops and terms of order $O(p^6)$ of ChPT. To avoid double counting we use the mesonic contribution to $f_+^{\rm mes}(0) = 1.0002 \pm 0.0022$ and the p^6 terms one $f_+(0)|_{p^6} = -0.016 \pm 0.0008$. Our final result is

$$|V_{us}| = 0.21715 \pm 0.0055. \tag{54}$$

In estimating the uncertainty we take into account the uncertainties arising from the structure-dependent emission

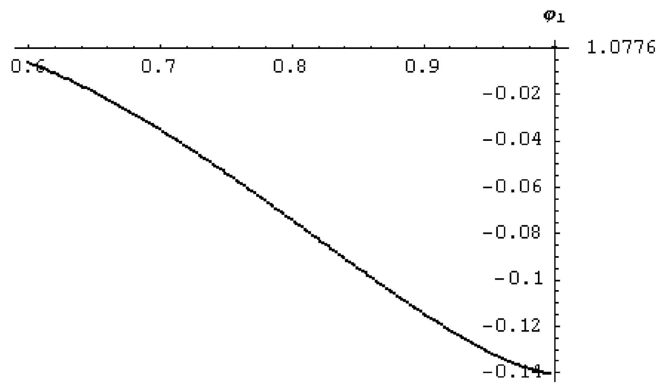

Fig. 5. Correction to pion spectrum, $\phi_1(z)$ (see (39))

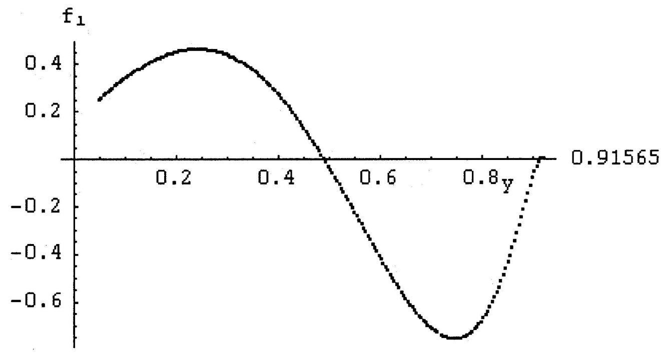

Fig. 6. Correction to positron spectrum, $f_1(y)$ (see (40))

±0.005, theoretical errors of order ±0.0003, the experimental error ±0.0022 and the ChPT error in the p^6 terms 0.0008.

In Tables 1 and 2 we give corrections to the distributions in the Dalitz plot $\mathrm{d}\Gamma/(\mathrm{d}y\mathrm{d}z) \sim a_0(y,z) + (\alpha/\pi)\,\Delta(y,z)$.

In Figs. 3–6 we illustrate the corrections to the pion and positron spectra. Here we see qualitative agreement for the positron spectrum and disagreement with the pion spectrum obtained by Ginsberg.

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Appendix A

Here we explain how to calculate $\delta_{\rm S}$, $\delta_{\rm C}$, $\delta_{\rm PLM}$ and how to group them into (19).

The contribution from emission of a soft real photon can be written in a standard form in terms of the classical currents:

Table 1. Correction to Dalitz-plot distribution $\Delta(y,z) = a_0(y,z)\delta(y,z) \times 10^3$ (see (2))

z/y	0.07	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85
1.025	3.83	4.37	3.71	2.01	−0.21	−2.47	−4.31	−5.11	−3.81
0.975		3.76	3.49	2.07	0.05	−2.07	−3.83	−4.61	−3.35
0.925			3.26	2.13	0.32	−1.67	−3.35	−4.11	−2.88
0.875			3.04	2.18	0.58	−1.26	−2.86	−3.60	−2.39
0.825				2.25	0.85	−0.86	−2.37	−3.08	−1.88
0.775					1.14	−0.41	−1.83	−2.51	−1.28
0.725					1.39	−0.04	−1.39	−2.03	−0.72
0.675						0.38	−0.86	−1.48	
0.625							−0.35	−0.89	
0.580								−0.23	

Table 2. Dalitz-plot distribution in Born approximation $a_0(y,z)$

z/y	0.07	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85
1.025	0.0084	0.069	0.126	0.163	0.181	0.178	0.156	0.114	0.051
0.975		0.026	0.089	0.131	0.153	0.156	0.139	0.101	0.0437
0.925			0.051	0.099	0.126	0.134	0.121	0.088	0.036
0.875			0.014	0.066	0.099	0.111	0.104	0.076	0.029
0.825				0.0337	0.071	0.089	0.086	0.064	0.021
0.775					0.043	0.066	0.069	0.051	0.014
0.725					0.016	0.044	0.051	0.039	0.006
0.675						0.021	0.034	0.026	
0.625							0.016	0.014	
0.580								0.003	

$$\delta_{\mathrm{S}} = -\frac{4\pi\alpha}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{2\omega}\left(\frac{p}{p\cdot q} - \frac{p_e}{p_e\cdot q}\right)^2\Bigg|_{\omega=\sqrt{\vec{q}^2+\lambda^2}<\Delta\epsilon}, \tag{55}$$

where λ is the fictitious mass of the photon. We use the following formulas:

$$\frac{1}{2\pi}\int \frac{\mathrm{d}^3q}{2\omega}\left(\frac{p}{p\cdot q}\right)^2 = \ln\left(\frac{2\Delta\epsilon}{\lambda}\right) - 1;$$
$$\frac{1}{2\pi}\int \frac{\mathrm{d}^3q}{2\omega}\left(\frac{p_e}{p_e\cdot q}\right)^2 = \ln\left(\frac{2\Delta\epsilon}{\lambda}\right) - \frac{1}{2}L_e;$$
$$\frac{1}{2\pi}\int \frac{\mathrm{d}^3q}{2\omega}\frac{2(p\cdot p_e)}{(p\cdot q)(p_e\cdot q)} = L_e \ln\left(\frac{2\Delta\epsilon}{\lambda}\right) - \frac{\pi^2}{6} - \frac{1}{4}L_e^2. \tag{56}$$

From them we obtain (16).

Consider now radiative corrections that arise from the emission of virtual photons (excluding SD virtual photons).

Feynman graphs containing a self-energy insertion to the positron and kaon Green functions (Fig. 1b,c) can be taken into account by introducing the wave function renormalization constants Z_e and Z_K: $M_0 \to M_0(Z_K Z_e)^{1/2}$. We use the expression for Z_e given in the textbooks [16]; the expression for Z_K is given in [15]. The result is (17).

Now consider the Feynman graph in which a virtual photon is emitted by a kaon and absorbed by a positron or by a W-boson in the intermediate state (Fig. 1d,e,f). This long distance contribution is calculated using a phenomenological model with point-like mesons as the relevant degrees of freedom. To calculate the contribution from the region $|k|^2 < \Lambda^2$ (Λ is the ultraviolet cutoff) we use the following expressions for the loop momenta scalar, vector, and tensor integrals:

$$\mathrm{Re}\int \frac{\mathrm{d}^4k}{\mathrm{i}\pi^2}\frac{1, k^\mu, k^2}{(k^2-\lambda^2)((k-p)^2-M^2)((k-p_e)^2-m_e^2)} = I, I^\mu, J. \tag{57}$$

A standard calculation yields

$$I = \frac{-1}{yM^2}\left\{\frac{1}{2}\ln\frac{M^2}{\lambda^2}L_e + \ln^2 y + \mathrm{Li}_2(1-y) - \frac{1}{4}\ln^2 r_e\right\};$$
$$I^\mu = \frac{-1}{yM^2}\left\{\frac{-y\ln y}{1-y}p^\mu + p_e^\mu\left(\frac{y\ln y}{1-y} + L_e\right)\right\};$$
$$J = L_\Lambda + \frac{y\ln y}{1-y} + 1, \tag{58}$$

where $L_\Lambda = \ln(\Lambda^2/M^2)$ and we omitted terms of the order of $O(m_e^2/M^2)$. As a result we obtain

$$\int \frac{\mathrm{d}^4k}{\mathrm{i}\pi^2}\frac{(1/4)\mathrm{Sp}p_\nu(p+p')(-\hat{p}_e+\hat{k})(2\hat{p}-\hat{k})p_e(p+p')}{(k^2-\lambda^2)((k-p)^2-M^2)((k-p_e)^2-m_e^2)}$$

$$= 2M^4 a_0(y,z)\left\{-L_\Lambda - \frac{1}{2}\ln^2 r_e - 2L_e + \ln\frac{M^2}{\lambda^2}L_e - 1 \right.$$
$$\left. +\ 2\ln^2 y + 2\ln y + 2\mathrm{Li}_2(1-y)\right\}. \qquad (59)$$

In a series of papers [7] Sirlin has conducted a detailed analysis of the UV behavior of the amplitudes of processes with hadrons in one-loop level. He showed that they are UV finite (if considered on the quark level), but the effective cutoff scale on the loop momenta is of the order M_W. For this reason we choose

$$L_\Lambda = \ln\frac{M_W^2}{M^2}.$$

The sum $\delta_\mathrm{S} + \delta_\mathrm{C} + \delta_\mathrm{PLM}$ yields (19).

Appendix B

The matrix element of the radiative K_{e3} decay

$$K^+(p) \to \pi^0(p') + e^+(p_e) + \nu(p_\nu) + \gamma(q), \qquad (60)$$

with terms up to $O(p^2)$ in CHPT [3–6] has the form

$$M^\mathrm{hard} = \frac{G_\mathrm{F}}{2} f_+ V^*_{us}\sqrt{4\pi\alpha}\,\bar{u}(p_\nu)Q^\mathrm{hard}_\mu(1+\gamma_5)v(p_e)\epsilon^\mu(q), \qquad (61)$$

where

$$Q^\mathrm{hard}_\mu = Q^e_\mu + Q^\pi_\mu + Q^\mathrm{SD}_\mu = Q^\mathrm{IB}_\mu + Q^\mathrm{SD}_\mu;$$
$$Q^\mathrm{IB}_\mu = (\hat{p}+\hat{p}')\left[\frac{(-\hat{p}_e - \hat{q} + m_e)\gamma_\mu}{2p_e q} + \frac{p_\mu}{pq}\right];$$
$$Q^\mathrm{SD}_\mu = \gamma_\nu R_{\mu\nu}, \qquad (62)$$

where the tensor $R_{\mu\nu}$ describes (see (4.17) in [4]) the structure-dependent emission (Fig. 1c). We have

$$R_{\mu\nu} = g_{\mu\nu} - \frac{q_\nu p_\mu}{pq}. \qquad (63)$$

Terms singular at $\chi = 2p_e q \to 0$ which provide a contribution containing a large logarithm L_e arise only from Q^e_μ. To extract the corresponding terms we introduce the 4-vector $v = (x/y)p_e - q$, and x is the energy fraction of the photon (9). Note that $v \to 0$ when $\chi \to 0$. Separating leading and non-leading terms and letting $f_+(t) = 1$, i.e. neglecting the form factor's momentum dependence, we obtain

$$\delta_\mathrm{H} = \frac{\mathrm{d}\Gamma^\mathrm{hard}}{\mathrm{d}\Gamma_0} = \frac{\alpha}{2\pi a_0(y,z)}\int\frac{\mathrm{d}x}{x}\int\frac{\mathrm{d}O_\gamma}{2\pi}T,$$
$$x > y\Delta, \qquad (64)$$

where

$$T = \frac{x^2}{8}\sum_\mathrm{spins}\left|\bar{u}(p_\nu)\left(Q^\mathrm{hard}_\mathrm{IB} + Q^\mathrm{hard}_\mathrm{SD}\right)(1+\gamma_5)v(p_e)\right|^2$$
$$= \frac{y a_0(x+y,z)}{x+y}\left[\frac{y^2+(x+y)^2}{y^2(1-\beta_e C_e)} - 2\frac{(1-\beta_e)(x+y)}{y(1-\beta_e C_e)^2}\right]$$
$$- \frac{y a_0(x+y,z)}{x+y} + \mathcal{P}. \qquad (65)$$

The quantity $\mathcal{P}$ contains some non-leading contributions from the IB part and the ones that arise from the structure-dependent part:

$$\mathcal{P} = \left(\frac{p_e q}{M^2}\left(\frac{p_\nu q}{M^2} + z - \frac{2y}{x+y}(1-x-y)\right)\right.$$
$$\left. + \frac{p'v}{M^2}\frac{y(2-x-y)}{x+y}\right)$$
$$\times\left(\frac{xM^2}{4yp_e q}(y^2+(x+y)^2) - 1\right)$$
$$- \frac{M^2x^2}{8p_e q}\left(T_v + \frac{2}{x}T_{1v}\right) - \frac{x^2}{8}(T_{RR} + 2T_R), \qquad (66)$$

with

$$T_v = \frac{1}{4M^4}\mathrm{Sp}(\hat{p}+\hat{p}')\hat{p}_\nu(\hat{p}+\hat{p}')\hat{v};$$
$$T_{1v} = \frac{1}{4M^6}\mathrm{Sp}(\hat{p}+\hat{p}')\hat{p}_\nu(\hat{p}+\hat{p}')\hat{v}\hat{p}\hat{p}_e;$$
$$T_{RR} = R_{\mu\lambda}R_{\mu\sigma}\frac{1}{4M^2}\mathrm{Sp}\hat{p}_\nu\gamma_\lambda\hat{p}_e\gamma_\sigma; \qquad (67)$$
$$T_R = R_{\mu\lambda}\frac{1}{4M^2}\mathrm{Sp}\hat{p}_\nu(\hat{p}+\hat{p}')\left[\frac{p_\mu}{pq} - \frac{(\hat{p}_e+\hat{q})\gamma_\mu}{\chi}\right]\hat{p}_e\gamma_\lambda.$$

To calculate these traces we use the following expressions for the scalar products of the 4-momenta (in units M):

$$p^2 = 1,\quad q^2 = 0,\quad p_\nu^2 = 0,\quad p'^2 = r_\pi,\quad p_e^2 = 0,$$
$$pp_e = \frac{y}{2},$$
$$pp' = \frac{z}{2},\quad pq = \frac{x}{2},\quad pp_\nu = \frac{1}{2}(2-y-z-x),$$
$$p'p_\nu = \frac{1}{2}(1-x-y-r_\pi+A_e),$$
$$p'q = \frac{1}{2}(x-A_e-A_\nu),$$
$$p'p_e = \frac{1}{2}(y-R(z)+A_\nu),\quad p_\nu q = \frac{1}{2}A_\nu,\quad p_e q = \frac{1}{2}A_e,$$
$$p_e p_\nu = \frac{1}{2}(R(z)-A_e-A_\nu),\quad pv = 0,\quad p_e v = -\frac{1}{2}A_e,$$
$$qv = \frac{1}{2}\frac{x}{y}A_e,\quad p'v = \frac{1}{2}\left(\frac{x+y}{y}\tilde{A}_\nu + A_e\right),$$
$$p_\nu v = -\frac{1}{2y}\left(xA_e + (x+y)\tilde{A}_\nu\right),$$
$$\tilde{A}_\nu = A_\nu - \frac{x}{x+y}R(z).$$

Three terms in the r.h.s. of (65) have a completely different behavior.

The first one corresponds to the kinematic region of collinear emission, when a photon is emitted along the positron's momentum. The relevant phase volume has essentially a three-particle form:

$$(\mathrm{d}\phi_4)^\mathrm{coll}$$

$$
\begin{aligned}
&= \left(\frac{\mathrm{d}^3 p_e}{2\epsilon_e}\frac{\mathrm{d}^3 q}{2\omega}\frac{\mathrm{d}^3 p'}{2\epsilon'}\mathrm{d}^4 p_\nu \delta(p_\nu^2)\right.\\
&\times \left.\delta^4(p-p_e-p_\nu-p'-q)\right)^{\mathrm{coll}}\\
&= M^4\frac{\pi^2}{64}\beta_\pi z\mathrm{d}zy\mathrm{d}yx\mathrm{d}x\mathrm{d}O_\gamma \mathrm{d}C_{e\pi}\\
&\times \delta\left(1-x-y-z+r_\pi+\frac{x+y}{y}\frac{zy}{2}(1-\beta_\pi C_{e\pi})+\frac{2p_e q}{M^2}\right)\\
&= \frac{y}{x+y}M^4\frac{\pi^2}{32}\mathrm{d}O_\gamma x\mathrm{d}x\mathrm{d}y\mathrm{d}z. \qquad (68)
\end{aligned}
$$

The limits of the photon's energy fraction variation are $y\Delta < x < b(z) - y$. The upper limit is imposed by the Born structure of the width in this kinematical region.

The second term corresponds to the contribution from emission by a kaon. The relevant kinematics is isotropic.

The kinematics of radiative kaon decay and the comparison of our and Ginsberg's approaches is given in Appendix F.

The third term corresponds to the rest of the contributions which contain neither collinear nor infrared singularities.

Performing the integration over the photon's phase volume provided y, z are in the D region, we obtain

$$
\begin{aligned}
&\int\frac{\mathrm{d}x}{x}\int\frac{\mathrm{d}O_\gamma}{2\pi}T\\
&= \int\limits_{y\Delta}^{b(z)-y}\frac{\mathrm{d}x}{x}\frac{y^2}{(y+x)^2}a_0(y+x,z)\\
&\times\left[\frac{y^2+(y+x)^2}{y^2}(L_e-1)+\frac{x^2}{y^2}\right] \qquad (69)\\
&-2\int\limits_{y\Delta}^{b(z)-y}\frac{\mathrm{d}x}{x}\left[a_0(y,z)+x\left(\frac{R(z)}{x+y}-y\right)\right]+\int\limits_0^{b(z)-y}\mathrm{d}x\mathcal{J},
\end{aligned}
$$

we obtain (27) with

$$
\begin{aligned}
\mathrm{Rphot}_{1D} &= \int\limits_0^{b(z)-y}\mathrm{d}x\left(\frac{R(z)}{x+y}-y\right)\\
&= R(z)\ln\frac{b(z)}{y} - y(b(z)-y); \qquad (70)\\
&(R(z)+y(2-z))\ln\frac{b(z)}{y}+\frac{1}{2}\mathrm{Rphot}_{2D}(y,z)\\
&= \int\limits_0^{b(z)-y}\mathrm{d}x\frac{xa_0(y+x,z)}{(y+x)^2}\\
&= -(R(z)+y(2-z))\ln\frac{b(z)}{y} \qquad (71)\\
&+\frac{1}{2}(b(z)-y)\left(2\frac{R(z)}{b(z)}+4-2z-b(z)+y\right),
\end{aligned}
$$

and

$$
\mathcal{J}(x,y,z) = \frac{1}{x}\int\frac{\mathrm{d}O_\gamma}{2\pi}\mathcal{P}. \qquad (72)
$$

One can check that the sum of RC arising from hard, soft and virtual photons does not depend on the auxiliary parameter Δ.

We note that the leading contribution from the hard part of the photon spectrum can be reproduced using the method of quasi-real electrons [10][1].

Now we concentrate on the contribution of the third term in the r.h.s. of (65).

To perform the integration over the phase volume of the final states it is convenient to use the following parameterization (see Appendix F):

$$
\begin{aligned}
\mathrm{d}\phi_4 &= \frac{\mathrm{d}^3p'\mathrm{d}^3p_e\mathrm{d}^3p_\nu\mathrm{d}^3q}{2\epsilon' 2E_e 2\epsilon_\nu 2\omega}\delta^4(p-p'-p_e-p_\nu-q)\\
&= \beta_\pi\frac{\pi^2}{16}M^4\mathrm{d}y\mathrm{d}zx\mathrm{d}x\frac{\mathrm{d}C_e\mathrm{d}C_\pi}{\sqrt{D}}, \qquad (73)
\end{aligned}
$$

with

$$
\begin{aligned}
D &= \beta_\pi^2(1-C^2-C_\pi^2-C_e^2+2CC_\pi C_e),\\
\beta_\pi &= \sqrt{1-\frac{4r_\pi}{z^2}}, \qquad (74)\\
C &= \cos(\vec{p}_e,\vec{p}'),\quad C_e=\cos(\vec{q},\vec{p}_e),\quad C_\pi=\cos(\vec{q},\vec{p}').
\end{aligned}
$$

The neutrino on-mass shell (NMS) condition provides the relation

$$
\begin{aligned}
1-\beta_\pi C &= \frac{2}{yz}\left[x+y+z-1-r_\pi-\frac{xz}{2}(1-\beta_\pi C_\pi)\right.\\
&\left.-\frac{xy}{2}(1-C_e)\right]. \qquad (75)
\end{aligned}
$$

For the aim of further integration of $\mathcal{P}$ over the angular variables we put it in the form

$$
\begin{aligned}
\mathcal{P} &= xP_1\frac{\tilde{A}_\nu}{A_e}+xP_2+P_3A_e+P_4A_\nu+P_5A_\nu A_e,\\
A_e &= \frac{xy}{2}(1-C_e),\\
A_\nu &= x-A_e-\frac{xz}{2}(1-\beta_\pi C_\pi), \qquad (76)
\end{aligned}
$$

and

$$
\begin{aligned}
P_1 &= \frac{y}{2}(1-x-y);\\
P_2 &= \frac{R(z)}{x+y}+\frac{1}{2}\left(z(2x+3y+1)+2x^2+4xy\right.\\
&\left.+3y^2-2x-3y-2\right);\\
P_3 &= 1-z-y-\frac{1}{2}x(x+y+z);\\
P_4 &= -1+x+y+\frac{1}{2}xy;\quad P_5=-1. \qquad (77)
\end{aligned}
$$

[1] The formula (10) in [10] should read

$$
\mathrm{d}\Gamma_b = \left.\frac{2\epsilon'\mathrm{d}^3\sigma_{0b}}{\mathrm{d}^3p'}\right|_{\vec{p}'=\vec{p}_3+\vec{k}}\mathrm{d}W_{\vec{p}_3+\vec{k}}(k)\frac{\mathrm{d}^3p_3}{2\epsilon_3}
$$

Angular integration can be performed explicitly; we have

$$\int \frac{\beta_\pi \mathrm{d}C_\pi}{\pi\sqrt{D}} = \frac{y}{\sqrt{A}},$$
$$\int \frac{\beta_\pi C_\pi \mathrm{d}C_\pi}{\pi\sqrt{D}} = \frac{y(x+y-yt)}{z\beta_\pi A^{3/2}} \times \left[2R(z) - (x+y)(2-z) + xyt\right], \tag{78}$$

with

$$A = (x+y)^2 - 2xyt, \quad t = 1 - C_e. \tag{79}$$

Performing the integration over C_π we have

$$\begin{aligned}\frac{1}{x}\int \frac{\mathrm{d}C_\pi \beta_\pi}{\pi\sqrt{D}}\mathcal{P} &= \frac{2y}{A^{3/2}}\left((y-x)\left(1-\frac{z}{2}-\frac{R}{x+y}\right)\right.\\
&- \left.\frac{1}{2}y\,(x+y-xt)\right)P_1\\
&+ \frac{y}{A^{1/2}}\left(P_2 + \frac{y}{2}tP_3\right)\\
&+ \left(P_4 + \frac{xy}{2}tP_5\right)\left\{\frac{y}{A^{1/2}}\left(1-\frac{z}{2}-\frac{y}{2}t\right)\right.\\
&+ \left.\frac{y}{A^{3/2}}(x+y-yt)\left(R-(x+y)\left(1-\frac{z}{2}\right)+\frac{xy}{2}t\right)\right\}.\end{aligned} \tag{80}$$

The following integrals are helpful in integrating the above expression. We define

$$I_n^m = \int_0^2 \frac{\mathrm{d}t t^m}{\sqrt{A^n}}, \quad m = 0,1,2,3; \quad n = 1,3. \tag{81}$$

Then

$$\begin{aligned}I_1^0 &= \frac{4}{\sigma}, \quad I_1^1 = \frac{8(x+y+\sigma)}{3\sigma^2},\\
I_1^2 &= \frac{16}{15\sigma^3}\left(3\sigma^2 + 3(x+y)\rho + 5(x+y)^2\right),\\
I_3^0 &= \frac{4}{\rho\sigma(x+y)}, \quad I_3^1 = \frac{8}{\rho\sigma^2},\\
I_3^2 &= \frac{16}{3\rho\sigma^3}\left(2(x+y)+\sigma\right),\\
I_3^3 &= \frac{32}{5\rho\sigma^4}\left(\sigma^2 + 2(x+y)\rho + 4(x+y)^2\right),\end{aligned} \tag{82}$$

where $\rho = |x-y|$ and $\sigma = x+y+\rho$.

The first term in $\mathrm{d}\Gamma^{\mathrm{hard}}$ together with the leading contributions from virtual and soft real photons was given in the form required by the RG approach (36).

The non-leading contributions, δ^{hard} from hard photon emission, include SD emission and IB of point-like mesons as well as the interference terms. It is free from infrared and mass singularities and given above (27) with

$$\mathcal{J}(x,y,z) = P_1R_1 + P_2yI_1^0 + P_3\frac{y^2}{2}I_1^1 + \frac{y}{2}P_4R_4 + \frac{xy^2}{4}P_5R_5, \tag{83}$$

and

$$\begin{aligned}R_1 &= \frac{y}{x+y}\\
&\times (y-x)\left((2-z)(x+y) - 2R(z)\right)I_3^0\\
&- y^2((x+y)I_3^0 - xI_3^1),\\
R_4 &= (2-z)\,I_1^0 - yI_1^1\\
&+ (2R(z) - (x+y)(2-z))((x+y)I_3^0 - yI_3^1)\\
&+ xy((x+y)I_3^1 - yI_3^2),\\
R_5 &= (2-z)\,I_1^1 - yI_1^2\\
&+ (2R(z) - (x+y)(2-z))((x+y)I_3^1 - yI_3^2)\\
&+ xy((x+y)I_3^2 - yI_3^3).\end{aligned}$$

Appendix C

The contribution to δ^{hard} from SD emission has the form

$$\delta^{\mathrm{hard}}_{\mathrm{SD}} = \frac{\alpha}{2\pi a_0(y,z)}\int_0^{b(z)} \mathrm{d}xJ^{\mathrm{SD}}(x,y,z), \tag{84}$$

where

$$J^{\mathrm{SD}}(x,y,z) = Q_1R_1 + yQ_2I_1^0 + \frac{y^2}{2}Q_3I_1^1 + \frac{y}{2}Q_4R_4 + \frac{xy^2}{4}Q_5R_5, \tag{85}$$

with R_i given in Appendix B and

$$\begin{aligned}Q_1 &= -\frac{1}{4}y\,(x+y),\\
Q_2 &= \frac{1}{4}\left[2x(x+2y+z-2) + 3y(y+z-2)\right],\\
Q_3 &= -\frac{1}{8}\left[-8 + (z+y)(4+3x) - 2x + 3x^2\right],\\
Q_4 &= \frac{1}{8}\left[4y + 4x + 3xy\right], \quad Q_5 = -\frac{3}{4}.\end{aligned} \tag{86}$$

The contribution to the total width has the form

$$\delta^{\mathrm{SD}} = \frac{\alpha}{2\pi}\frac{\int\int \mathrm{d}y\mathrm{d}z\left(1+\frac{\lambda_+}{r_\pi}R(z)\right)^2 \int_0^N \mathrm{d}xJ^{\mathrm{SD}}(x,y,z)}{\int\int \mathrm{d}y\mathrm{d}z a_0(y,z)\left(1+\frac{\lambda_+}{r_\pi}R(z)\right)^2}. \tag{87}$$

Numerical estimation gives

$$\delta^{\mathrm{SD}} = -0.005. \tag{88}$$

Appendix D

The function Ψ, defined by

$$\Psi(y,z) = \int_{b_-(z)}^{b(z)} \frac{\mathrm{d}t}{t}a_0(t,z)P^{(1)}\left(\frac{y}{t}\right), \tag{89}$$

contains a restriction on the domain of integration, namely t exceeding y or being equal to it, which is implied by the kernel $P^{(1)}(y/t)$. Explicit calculations give

$$\begin{aligned}\Psi_<(y,z) &= \int\limits_{b_-(z)}^{b(z)} \frac{\mathrm{d}t}{t} a_0(t,z)\frac{y^2+t^2}{t(t-y)} \\ &= [R(z)-y(2-z)]\ln\frac{b(z)}{b_-(z)} \\ &+ 2a_0(y,z)\ln\frac{b(z)-y}{b_-(z)-y} \\ &+ \frac{1}{2}(b(z)^2-b_-(z)^2), \\ &\quad 2\sqrt{r_e}<y<1-\sqrt{r_\pi}, \quad \Psi_<(y,z)=0, \\ &\quad y>1-\sqrt{r_\pi},\end{aligned} \tag{90}$$

and

$$\begin{aligned}\Psi_>(y,z) &= \int\limits_{y}^{b(z)} \frac{\mathrm{d}t}{t} a_0(t,z)P^{(1)}\left(\frac{y}{t}\right) \\ &= a_0(y,z)\left[2\ln\frac{b(z)-y}{y}+\frac{3}{2}\right] \\ &- \frac{1}{2}(b(z)^2-y^2)+(b(z)-y)(2-y-z+b_-(z)) \\ &+ [R(z)-y(2-z)]\ln\frac{b(z)}{y}.\end{aligned} \tag{91}$$

One can convince oneself of the validity of the relations

$$j_0(y) = \int\limits_{2\sqrt{r_\pi}}^{c(y)} \mathrm{d}z\Psi_<(y,z) + \int\limits_{c(y)}^{1+r_\pi} \mathrm{d}z\Psi_>(y,z) \tag{92}$$

and

$$\int\limits_{0}^{b_-(z)} \mathrm{d}y\Psi_<(y,z) + \int\limits_{b_-(z)}^{b(z)} \mathrm{d}y\Psi_>(y,z) = 0. \tag{93}$$

The last relation demonstrates the KLN cancellation for the pion spectrum obtained by integration of the corrections over y in the interval $0<y<b(z)$.

The explicit expressions for $j_1(y)$ and $j_2(y)$ are (for $j_0(y)$ see (43))

$$\begin{aligned}j_1(y) &= \frac{y^3(1-r_\pi-y)^3}{3(1-y)^2}\left(2\ln\frac{1-r_\pi-y}{y}+\frac{3}{2}\right) \\ &+ \frac{r_\pi^2}{3(1-y)^2}\Big[3(1-y)(1+y^2) \\ &+ r_\pi(y^3+3y-2)\Big]\ln\frac{1-y}{r_\pi} \\ &- \frac{1-r_\pi-y}{36(1-y)^2}\Big[(1-y)^2\Big(43y^3-15y^2-3y-1 \\ &+ r_\pi(83y^2+26y+11)+3r_\pi^3\Big) \\ &+ r_\pi^2(31y^3-15y^2-39y+47)\Big],\end{aligned} \tag{94}$$

$$\begin{aligned}j_2(y) &= \frac{y^4(1-r_\pi-y)^4}{12(1-y)^3}\left(2\ln\frac{1-r_\pi-y}{y}+\frac{3}{2}\right) \\ &+ \frac{r_\pi^2}{12(1-y)^3}\ln\frac{1-y}{r_\pi}\Big[6(1+y^2)(1-y)^2 \\ &- 4r_\pi(1-y)(2y^3-y^2+4y-3) \\ &+ r_\pi^2(y^4+6y^2-8y+3)\Big] \\ &+ \frac{1-r_\pi-y}{720(1-y)^3} \\ &\times\Big[-(1-y)^3(247y^4-88y^3-28y^2-8y-3) \\ &- r_\pi(1-y)^3(733y^3+341y^2+129y+57) \\ &- r_\pi^2(1-y) \\ &\times(707y^4-808y^3+212y^2-408y+717) \\ &+ r_\pi^3(173y^4-72y^3-492y^2+1048y-477) \\ &- 12r_\pi^4(1-y)^3\Big].\end{aligned} \tag{95}$$

Appendix E

Now follws a collection of the relevant formulae.

The Dalitz-plot distribution in the region D:

$$\begin{aligned}&\frac{1}{\mathcal{C}S_{\mathrm{EW}}}\frac{\mathrm{d}\Gamma}{\mathrm{d}y\mathrm{d}z} \\ &= \left(1+\lambda_+\frac{t}{m_\pi^2}\right)^2\Bigg(a_0(y,z) \\ &+ \frac{\alpha}{\pi}\left[\frac{1}{2}(L_e-1)\Psi_>(y,z)+a_0(y,z)Z_1+\frac{1}{2}Z_2\right]\Bigg), \\ Z_1 &= \frac{3}{4}-\frac{\pi^2}{6}-\frac{3}{2}\ln y-\ln((b(z)-y)/y) \\ &- \mathrm{Li}_2(1-y).\end{aligned} \tag{96}$$

The function Z_2 is defined in (28). The correction to the total width (we include the contribution of the region outside the region D), $\Gamma=\Gamma_0(1+\delta)$:

$$\begin{aligned}1+\delta &= S_{\mathrm{EW}}+\frac{\alpha}{\pi}\frac{1}{\int\int \mathrm{d}z\mathrm{d}y a_0(y,z)\left(1+\frac{\lambda_+}{r_\pi}R(z)\right)^2} \\ &\Bigg[\int\limits_0^{1-r_\pi} I(y)\ln y\mathrm{d}y + \int\limits_{2\sqrt{r_\pi}}^{1+r_\pi}\mathrm{d}z\left(1+\frac{\lambda_+}{r_\pi}R(z)\right)^2 \\ &\Bigg[\int\limits_0^{b_-(z)}\mathrm{d}y\left[-a_0(y,z)\ln\frac{b(z)-y}{b_-(z)-y}+(1/2)\tilde{Z}_2(y,z)\right] \\ &+ \int\limits_{b_-(z)}^{b(z)}\mathrm{d}y[a_0(y,z)Z_1+(1/2)Z_2]\Bigg]\Bigg],\end{aligned} \tag{97}$$

with

$$\begin{aligned}\tilde{Z}_2(y,z) &= \mathrm{Rphot}_{2A}(y,z) - 2\mathrm{Rphot}_{1A}(y,z)\\ &\quad+\int_{b_-(z)-y}^{b(z)-y} \mathrm{d}x \mathcal{J}(x,y,z);\\ \mathrm{Rphot}_{1A}(y,z) &= R(z)\ln\frac{b(z)}{b_-(z)} - y(b(z)-b_-(z))\\ \mathrm{Rphot}_{2A}(y,z) &= \int_{b_-(z)-y}^{b(z)-y}\frac{\mathrm{d}xx}{(x+y)^2}a_0(x+y,z)\\ &= (b(z)-b_-(z))\left(1-\frac{z}{2}+2y\right)\\ &\quad-(y(2-z)+R(z))\ln\frac{b(z)}{b_-(z)}. \qquad (98)\end{aligned}$$

The expression in big square brackets on the right-hand side of (94) can be put in the form

$$\int\limits_{2\sqrt{r_\pi}}^{1+r_\pi}\phi_1(z)\mathrm{d}z = \int\limits_{2\sqrt{r_e}}^{1-r_\pi} f_1(y)\mathrm{d}y = -0.035, \qquad (99)$$

which results in $\delta = 0.02$. For the aim of comparison with Ginsberg's result we must put here

$$\lambda_+ = 0, \quad I(y) = j_0(y), \quad M_W = M_p; \qquad (100)$$

as was mentioned above we have reasonable agreement with the Ginsberg results. For the inclusive set-up of the experiment (the energy fraction of the positron is not measured) we have for the pion energy spectrum given above (41). When we restrict ourselves only by the region D the spectrum becomes dependent on $\ln(1/r_e)$:

$$\begin{aligned}&\frac{1}{\mathcal{C}S_{\mathrm{EW}}}\frac{\mathrm{d}\Gamma}{\mathrm{d}z}\\ &= \Bigg\{\phi_0(z) + \frac{\alpha}{\pi}\Bigg[(1/2)P(z)(\ln(1/r_e)-1)\\ &+\int\limits_{b_-(z)}^{b(z)}\mathrm{d}y\left[\Psi_>(y,z)\ln y + a_0(y,z)Z_1 + \frac{1}{2}Z_2\right]\Bigg]\Bigg\}\\ &\times\left(1+\frac{\lambda_+}{r_\pi}R(z)\right)^2, \qquad (101)\end{aligned}$$

with

$$\begin{aligned}P(z) &= \frac{1}{6}b_-(z)^2(3b(z)+b_-(z))\ln\frac{b(z)}{b_-(z)}\\ &\quad+\frac{1}{3}(b(z)-b_-(z))^3\ln\frac{b(z)-b_-(z)}{b(z)}\\ &\quad-\frac{1}{6}b_-(z)(b(z)-b_-(z))(3b_-(z)+b(z)). \qquad (102)\end{aligned}$$

Appendix F

Our approach to the study of the radiative kaon decay has an advantage compared to the one used by Ginsberg – it has a simple interpretation of electron mass singularities based on the Drell–Yan picture. The approach of [2] to the study of non-collinear kinematics is more transparent than ours. We remind the reader of some some topics of [2]. One can introduce the missing mass square variable

$$\begin{aligned}l &= (p_\nu+k)^2/M^2 = A_\nu\\ &= (M-E_\pi-E_e)^2/M^2 - (\vec{p}_\pi+\vec{p}_e)^2/M^2; \quad (103)\end{aligned}$$

the limits of this quantity variation at fixed y,z are put by the last term: for collinear or anticollinear kinematics of pion and positron 3-momenta. Being expressed in terms of y,z they are (we consider the general point of a Dalitz plot and omit the positron mass dependence):

$$0 < l < b_-(z)(b(z)-y), \qquad (104)$$

for the y,z in the D region and

$$b(z)(b_-(z)-y) < l < b_-(z)(b(z)-y), \qquad (105)$$

for the case when they are in the region A outside D:

$$0 < y < b_-(z), \quad 2\sqrt{r_\pi} < z < 1+r_\pi. \qquad (106)$$

For our approach with separating the case of soft and hard photon emission we must modify the lower bound for l in the region D. This can be done using another representation of l:

$$l = x[1-(y/2)(1-C_e)-(z/2)(1-\beta C_\pi)], \qquad (107)$$

with C_e, C_π the cosine of the angles between the photon 3-momentum and the positron and pion ones, and $\beta = (1-m^2/E_\pi^2)^{1/2}$ is the pion velocity. The maximum of this quantity is $b(z)$. Taking this into account we obtain for the region of the hard photon

$$x > 2\Delta\varepsilon/M = y\Delta, \Delta = \Delta\varepsilon/E_e << 1, \qquad (108)$$

for the region D:

$$\begin{aligned}y\Delta < x < b(z)-y,\\ yb(z)\Delta < l < b_-(z)(b(z)-y); \qquad (109)\end{aligned}$$

and for region A:

$$\begin{aligned}b_-(z)-y < x < b(z)-y,\\ b(z)(b_-(z)-y) < l < b_-(z)(b(z)-y). \qquad (110)\end{aligned}$$

In particular for the collinear case we must choose $C_e = 1; C_\pi = -1$, which corresponds to $x+y < b(z)$. Let us infer this condition using the NMS condition:

$$\begin{aligned}&(P_k-p_e-p_\pi-k)^2/M^2\\ &= R(z)-x-y+(xy/2)(1-C_e)\\ &+(xz/2)(1-\beta C_\pi)+(yz/2)(1-\beta C_{e\pi}) = 0. \qquad (111)\end{aligned}$$

In the collinear case we have $C_e = 1; C_\pi = C_{e\pi}$. From the NMS condition we obtain $1 - \beta C_\pi = (2/z(x+y))(x+y-R(z))$. Using this value we obtain $l_{\rm coll} = R(z)x/(x+y)$. Using further the relation $R(z) = b(z)b_-(z)$ we obtain again $x < b(z) - y$ in the case of emission along the positron.

Comparing the phase volumes in the general case calculated in our approach using the NMS condition with the approach of [2], we obtain the relation

$$\int x\mathrm{d}x \int \frac{\mathrm{d}O_\gamma}{4\pi} = \int \mathrm{d}l \int \mathrm{d}\gamma, \tag{112}$$
$$\int \mathrm{d}\gamma = \int \frac{\mathrm{d}^3 p_\nu}{E_\nu}\frac{\mathrm{d}^3 k}{k_0}\frac{\delta^4(P-p_\nu-k)}{2\pi}.$$

The non-leading contribution arising from hard photon emission considered above is

$$I_{\rm IB} = \int \frac{\mathrm{d}x}{x}\int \frac{\mathrm{d}O_\gamma}{4\pi}\mathcal{P}_{\rm IB}, \tag{113}$$

with

$$\begin{aligned}
\mathcal{P}_{\rm IB} &= xG_1\frac{\tilde{A}_\nu}{A_e} + xG_2 + G_3A_e + G_4A_\nu + G_5A_eA_\nu,\\
G_1 &= \frac{y}{4}(2-y-x),\\
G_2 &= \frac{R(z)}{2(x+y)} + \frac{x^2}{2} + \frac{1}{2}x(z+2y)\\
&\quad + \frac{1}{4}(2z + 3y(y+z)) - 1;\\
G_3 &= -\frac{1}{8}x^2 - \frac{1}{8}x(2+z+y) - \frac{1}{2}(y+z);\\
G_4 &= \frac{1}{8}x(4+y) + \frac{1}{8}y - 1;\\
G_5 &= -\frac{1}{4}
\end{aligned} \tag{114}$$

(note that $G_i + Q_i = P_i$; see Appendix B and C) can be transformed to the form

$$\begin{aligned}
I_{\rm IB} = (1/4)\int \mathrm{d}l\Bigg[&4 - 2y - 4z - (1/4)R(z) + (1/4)l\\
&+ y\ln\frac{(R(z)-l)^2}{l} - 2\ln\frac{y^2R(z)^2}{l(l+y(2-z))}\\
&+ [z + (3/2)y(y+z) - 2 + (1/4)l(4+y)]I_{10}\\
&- (1/2)I_{1-1} - ((1/2)l + y + z)I_{2-1} + I_z\Bigg].
\end{aligned} \tag{115}$$

Here we use the list of integrals obtained in [2]:

$$\begin{aligned}
I_{mn} &= \int \mathrm{d}\gamma \frac{1}{(kP_K/M^2)^m(kp_e/M^2)^n};\\
I_{10} &= \frac{2}{s}\ln\frac{2-y-z+s}{2-y-z-s};\quad I_{20} = 4/l;\quad I_{00} = 1;\\
I_{-1,0} &= (2-y-z)/2;\quad I_{11} = \frac{4}{yl}\ln\frac{y^2}{l};\\
I_{01} &= \frac{2}{R(z)-l}\ln\frac{(R(z)-l)^2}{lr_e};\\
I_{1-1} &= \frac{R(z)(2-y-z)-(2+y-z)l}{s^2}\\
&\quad + \frac{2l(y(2-y-z)-2R(z)+2l)}{s^3}\\
&\quad \times \ln\frac{2-y-z+s}{2-y-z-s};\\
I_{2-1} &= \frac{2(y(2-y-z)+2l-2R(z))}{s^2}\\
&\quad + \frac{R(z)(2-y-z)-(2+y-z)l}{s^3}\\
&\quad \times \ln\frac{2-y-z+s}{2-y-z-s},\\
s &= \sqrt{(2-y-z)^2-4l}.
\end{aligned} \tag{116}$$

Besides these we need two additional ones:

$$\begin{aligned}
I_e &= \int \mathrm{d}\gamma\frac{1}{(kp_e/M^2)(2(kP_K/M^2)+y)}\\
&= \frac{2}{yR(z)}\ln\frac{y^2R(z)^2}{l(l+y(2-z))r_e};\\
I_z &= \int \mathrm{d}\gamma\frac{1}{(kP_K/M^2)(2(kP_K/M^2)+y)}\\
&= \frac{4}{ys}\ln\frac{2l+ys+y(2-y-z)}{2l+ys-y(2-y-z)}.
\end{aligned} \tag{117}$$

One can see the cancellation of mass singularities (terms containing $\ln(1/r_e)$) in the expression $I_{\rm IB}$.

Numerical calculations are in agreement (within a few percent) with this and the expressions given above.

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